Exercise 90

The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

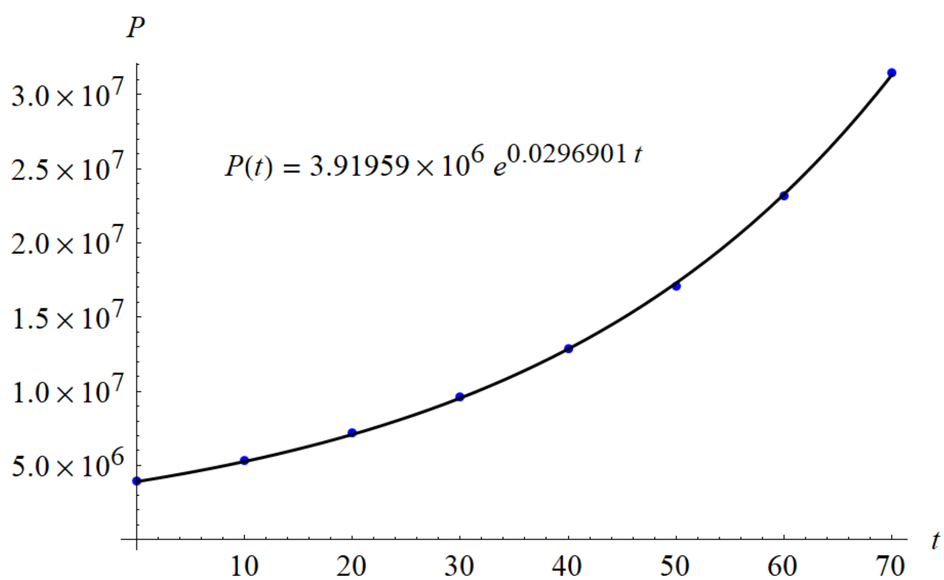
(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.

(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).

(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

Solution

Plot the given data in Mathematica and use the FindFit function to determine the function $Q(t) = Ae^{kt}$ that best fits the data. Here t is the number of years after 1790.



Estimate the rate of population growth in 1800 by averaging the slopes of the nearest secant lines.

$$\left.\frac{dP}{dt}\right|_{t=10} \approx \left.\overline{\frac{\Delta P}{\Delta t}}\right|_{t=10} = \frac{\frac{P(10)-P(0)}{10-0} + \frac{P(20)-P(10)}{20-10}}{2} = \frac{\frac{5\,308\,000-3\,929\,000}{10} + \frac{7\,240\,000-5\,308\,000}{10}}{2} = 165\,550$$

Estimate the rate of population growth in 1850 by averaging the slopes of the nearest secant lines.

$$\left.\frac{dP}{dt}\right|_{t=60} \approx \left.\overline{\frac{\Delta P}{\Delta t}}\right|_{t=60} = \frac{\frac{P(60)-P(50)}{60-50} + \frac{P(70)-P(60)}{70-60}}{2} = \frac{\frac{23\,192\,000-17\,063\,000}{10} + \frac{31\,443\,000-23\,192\,000}{10}}{2} = 719\,000$$

Differentiate the population model.

$$\begin{aligned} P'(t) &= \frac{dP}{dt} \\ &= \frac{d}{dt}[(3.91959 \times 10^6)e^{0.0296901t}] \\ &= (3.91959 \times 10^6)\frac{d}{dt}(e^{0.0296901t}) \\ &= (3.91959 \times 10^6)(e^{0.0296901t}) \cdot \frac{d}{dt}(0.0296901t) \\ &= (3.91959 \times 10^6)(e^{0.0296901t}) \cdot (0.0296901) \\ &\approx 116\,373e^{0.0296901t} \end{aligned}$$

Evaluate it at $t = 10$ and $t = 60$.

$$P'(10) \approx 156\,601$$

$$P'(60) \approx 691\,047$$

Plug in $t = 80$ to $P(t)$ to determine the population in 1870.

$$P(80) = (3.91959 \times 10^6)e^{0.0296901(80)} \approx 42\,148\,400$$

This is an overestimation; the discrepancy comes from the fact that the human population doesn't actually grow exponentially because resources and space are limited.